MPHYCC-3 P_01

Unit -5

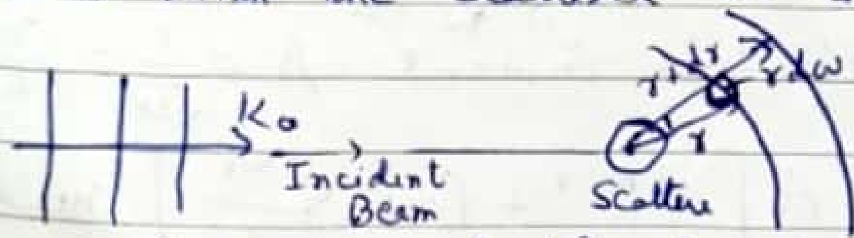
Scattering in Q. M.

Quantum particles are described through wave functions. A monoenergetic incident beam of quantum particles moving towards scatterer may be represented by free particle wave fn. like

$$\phi_k(r) = e^{ik.r} \quad \text{where } k = \frac{\sqrt{2mE}}{\hbar} = \frac{2\pi}{\lambda}$$

$$p = \sqrt{2mE}$$

when classical plane wave ~~reaches~~ reach a scatterer, these are scattered or diffracted in all possible directions in the form of spherical waves with the scatterer as their centre



Elementary Volume $= r^2\, dr\, d\omega$

The wave fn. ψ_k in $\left\{-\frac{\hbar^2}{2m}\nabla^2 + V_{(r)}\right\}\psi_k(r) = E\,\psi_k(r)$

may be written as $\psi_k = \psi_k(r, \theta, \phi)$

The scattering is determined by the asymptotic form of ψ_k in the region $V = 0$ (for $r \to \infty$).

The outgoing scattered wave is given by

$$\psi_k(r, \theta, \phi) \underset{r\to\infty}{=} e^{ik.r} + f(\omega)\frac{e^{ik.r}}{r}$$

With J = current density vector $= \frac{\hbar}{2im}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right)$

The Hamiltonian may be visualized as $H = H^0 + H'$ then S-eqn is

$$(H^0 + H')\psi = E\psi \quad - (6)$$

for $V(r) << E$
$V(r)$ may be regarded as perturbation Operator.

The unperturbed S-eqn. is

$$\left(\frac{-\hbar^2}{2m}\nabla^2 - E\right) e^{ik.r} = 0$$

Using above relation,
So, the S-eqn. (1) is now written as,

$$\left(\frac{-\hbar^2}{2m}\nabla^2 - E\right)\psi_s = -V(r)\left[e^{ik.r} + \psi_s\right] \quad (7)$$

$$= -V(r)\,\psi(r)$$

Multiply by $\frac{2m}{\hbar^2}$,

$$\Rightarrow (\nabla^2 + k^2)\psi_s = \frac{2m}{\hbar^2} V(r)\,\psi(r) \quad (8)$$

where $\frac{2mE}{\hbar^2} = k^2$

General Formulation of Scattering Theory :

$$H\psi = E\psi$$

The S-eqn. [for Central potential $V(r)$ is

$$\left\{-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right\}\psi = E\psi \quad \text{— (1)}$$

With complete time-dependent Soln. written as

$$\psi_{(r,t)} = \psi_{(r)} \cdot e^{-\frac{iE}{\hbar}t} \quad \text{— (2)}$$

Scattered wave

Incident wave

$$= \left[e^{ik.r} + \psi_{s(r)}\right] e^{-\frac{iE}{\hbar}t} \quad \text{— (3)}$$

where $\psi_s(r) = f(\theta,\phi)\frac{e^{ikr}}{r} + g(\theta,\phi)\frac{e^{-ikr}}{r}$ — (4)

outgoing Scattered wave

Incoming Scattered wave (doesnot exist in most of the Physical problem)

The stationary state Soln. is

$$\psi_{(r)} = e^{ik.r} + \psi_s \quad \text{— (5)}$$

No. of Scattered per unit volume is termed as

Density of scattered particles

$$\rho_s = \left| f(\omega) \frac{e^{ikr}}{r} \right|^2 = \frac{1}{r^2} |f(\omega)|^2$$

No of Particles in elementary volume

$$N_s = \rho_s r^2 \, dr \, d\omega = \frac{|f(\omega)|^2}{r^2} r^2 \, dr \, d\omega$$

No. of Scattered particle per unit time

$$\frac{dN_s}{dt} = |f(\omega)|^2 d\omega \cdot \underbrace{\left(\frac{dr}{dt}\right)}_{v} = |f(\omega)|^2 \frac{\hbar K}{m} \cdot d\omega$$

$\sigma(\omega)$ = Scattering Cross-Section (i.e. No. of particles scattered in unit Solid angle $d\omega$ per unit time)

$$= J \sigma(\omega) d\omega = \frac{\hbar K}{m} \sigma(\omega) \, d\omega$$

$$= |f(\omega)|^2$$ where $f(\omega)$ = Scattering Amplitude

From normalliscation Condition $\int \psi_k^* \psi_k \, d\tau = 1$ we get Normalliscation Constant $A = \frac{1}{\sqrt{v}} = \left(\frac{m}{\hbar K}\right)^{1/2}$

So, $\psi_k = \underset{r \to \infty}{} A \left[e^{iK \cdot r} + f(\omega) \frac{e^{iKr}}{r} \right]$ needs modifications as the outgoing wave function can't be only sum of current incident plane wave and scattered wave. It is over simplification on the interference b/w incident & scattered is completed ignored. The initial consideration of representation of incident wave as $\psi_k(r) \, e^{i\frac{E}{\hbar}t}$ is an idealisation and for real physical situation, concept of wave packets need to be incorporated.

$$\Rightarrow (\nabla^2 + k^2)\,\psi_s = -4\pi\, \rho(r) \quad -(9)$$

where $\frac{2m}{\hbar^2} V(r)\, \psi(r) = -4\pi \rho(r)$

$\rho(r)$ is regarded as source density for divergent sp. waves.

Eqn. (9) may be solved by using principle of superposition. If ψ_{s1} and ψ_{s2} are solutions of eqn. (9) belonging to density functions $\rho(r_1)$ and $\rho(r_2)$ and satisfying

$$\left.\begin{aligned} \psi_{s1} &= f_1 \cdot \frac{e^{ikr}}{r} \\ \psi_{s2} &= f_2 \cdot \frac{e^{ikr}}{r} \end{aligned}\right\} \quad -(10)$$

then fun. $\psi_s = \psi_{s1} + \psi_{s2}$ is also a soln. of eqn (9) belonging to $\rho(r) = \rho_1(r) + \rho_2(r)$ such that

$$\psi_s = f \cdot \frac{e^{ik \cdot r}}{r} \quad \text{where } f = f_1 + f_2$$

$$-(11)$$

Using Principle of Superposition, a soln. of Eq. (9) can be found by adding Soln. for simple point source of unit strength, expressed by in terms of arbitary density $\rho(r)$,

$$\rho(r) = \int \delta(r-r')\, \rho(r')\, dr' \quad (12)$$

ψ_s can be expressed in terms of $\rho(r)$ and eqn. (9) can be expressed in terms of Green's function

$$G(r,r') = \frac{\exp(ik|r-r'|)}{|r-r'|}$$

as.

$$\boxed{(\nabla^2 + K^2)\, G(r,r') = -4\pi\, \delta(r-r')}$$

If $G(r,r')$ is asymptotic to a fun. of r of the form $\psi_s = f \cdot \frac{e^{ikr}}{r}$ then soln. of the Scattering problem for the density $\rho(r)$ is giv

$$\psi_s = \int G(r, r')\, f(r')\, d\tau'$$

$$\psi_s = -\int G(r, r')\, V(r')\, \psi(r')\, d\tau'$$

$$\psi_{s\,(r \to \infty)} = -\frac{m}{2\pi \hbar^2}\left(\frac{e^{ikr}}{r}\right) \int e^{-ik' \cdot r'} \cdot V(r')\, \psi(r')\, d\tau'$$

$$= f(\theta, \phi) \cdot \left(\frac{e^{ikr}}{r}\right)$$

This gives Scattering Amplitude as

$$f(\theta, \phi) = -\frac{m}{2\pi \hbar^2} \int e^{-ik' \cdot r'}\, V(r')\, \psi(r')\, d\tau'$$

and Scattering Cross-Section as

$$\sigma(\theta, \phi) = |f(\theta, \phi)|^2 = \left(\frac{m}{2\pi \hbar^2}\right)^2 \left| \int e^{-ik' \cdot r'} \cdot V(r')\, \psi(r')\, d\tau' \right|^2$$